Math 53 – Worksheet 3

Partial Differentiation and Chain Rule GSI: Oltman, (3/2/20)

Problem 1 (**Laplace Equation in Polar Coordinates**). *The goal of this problem is to use the chain rule to transform the differential operator $\partial_{xx} + \partial_{yy}$ to polar coordinates.*

1. *Let $f = f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$. Note that $f_x = f_r r_x + f_\theta \theta_x$, write a similar expression for f_y.*
2. *We can differentiate the f_x again to get $f_{xx} = (f_{rr}r_x + f_{r\theta}\theta_x)r_x + f_r r_{xx} + (f_{\theta r}r_x + f_{\theta\theta}\theta_r)\theta_x + f_\theta \theta_{xx}$. Write the same expression for f_{yy}.*
3. *Add together f_{yy} and f_{xx} and simplify (one term should be $f_r(r_{xx} + r_{yy})$.*
4. *Use $r^2 = x^2 + y^2$ to get expressions for $r_{xx} + r_{yy}$ and $r_x^2 + r_y^2$ (the second should equal 1) .*
5. *Use $\tan\theta = \dfrac{y}{x}$ to get expressions for θ_x, θ_y and second derivatives, should get $\theta_x^2 + \theta_y^2 = 1/r^2$*
6. *Plug in everything to hopefully get $\partial_{xx} + \partial_{yy} = \partial_{rr} + \dfrac{1}{r}\partial_r + \dfrac{1}{r^2}\partial_{\theta\theta}$*
7. *Verify that $y/(x^2 + y^2)$ solves the laplace equation (use polar coordinates and use the above).*

Problem 2. *Let $T(x, y) = x^2 e^y - xy^3$ with $x(t) = \cos(t)$ and $y(t) = \sin(t)$, find $\dfrac{dT}{dt}$ by the chain rule and by substituting t in and differentiating normally.*

Problem 3. *Find the point in $\mathbb{R}^3$ where the $z-$axis intersects the plane that is tangent ot the graph of $z = e^{x-y}$ at $(1, 1, 1)$*

Problem 4. *Suppose that the energy of our system is given by $E(t, x)$ where t is time and x is a particle in our system, which is itself a function of time and it's mood $x = x(t, m)$. Use the chain rule to write $\dfrac{\partial E}{\partial t}$. Do you see the notational issue with partial differentiation?*

Problem 5. *Show that $f(x, y) = x^2 + y^2$ is differentiable by showing that*

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to 0$

Problem 6 (**Challenge**). *Let $f = f(x, y)$ be a infinitely differentiable function such that $f(x, y) \leq 0$ for all $x^2 + y^2 < r$ for a fixed r and $f(x, y) = 0$ for $x^2 + y^2 = 1$. So on a disc of radius r, f is bounded by the values on its boundary. Prove that the directional derivative of f in the outward normal direction of the circle of radius r, at any point on the circle, is greater than or equal to 0.*